ON THE NUMBER OF ALGEBRAICALLY INDEPENDENT POINCARE-LIAPUNOV CONSTANTS

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Abstract. In this paper an upper bound for the number of algebraically independent Poincaré-Liapunov constants in a certain basis for planar polynomial differential systems is given. Finally, it is conjectured that an upper bound for the number of functionally independent Poincaré-Liapunov quantities would be $m^2 + 3m - 7$ where $m$ is the degree of the polynomial differential system. Moreover, the computational problems which appear in the computation of the Poincaré-Liapunov constants and in the determination of the center cases are also discussed.

1. Introduction

One of the most interesting and difficult problems in the theory of planar differential equations is the control of the number of limit cycles that a differential equation or a family of differential equations can have. There exists different methods to produce limit cycles: the limit cycles which bifurcate from a singular point, from a center, from a homoclinic or heteroclinic orbit and finally the limit cycles which bifurcate from the infinity. To detect these different types of bifurcations of limit cycles there exists different methods: the first one is based on the study of the Poincaré return map, the Poincaré-Melnikov integral method, the Abelian integral method, the averaging method, and the last one uses the inverse integrating factor. When the first is used to detect the small limit cycles from a singular point is called Degenerate Hopf bifurcation, see [4] and [21]. It can be also used to detect limit cycles which bifurcated from a center, see [6, 10]. The second and third methods are based in the study of small perturbations of Hamiltonian

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systems, and in fact in the plane are essentially equivalent, see [6], Section 6 of Chapter 4 of [17] and Section 5 of Chapter 6 of [3].

Other classical problem in the qualitative theory of planar analytic differential systems is to characterize the local phase portrait at an isolated singular point. This problem has been solved except if the singular point is of focus–center type, see [1, 2, 12]. Recall that a singular point is said to be of focus–center type if it is either a focus or a center. The problem of distinguishing between a center or a focus is called the center problem. Of course, if the linear part of the singular point in nondegenerate (i.e. its determinant does not vanishes) the characterization is well known, see [18, 19]. If an analytic system has a nondegenerate singular point of focus type at the origin, then after a linear change of variables and a rescaling of the time variable it can be written into the form:

\[
\begin{align*}
\dot{x} &= \lambda x - y + \tilde{X}(x, y), \\
\dot{y} &= x + \lambda y + \tilde{Y}(x, y),
\end{align*}
\]

where \(\tilde{X}(x, y)\) and \(\tilde{Y}(x, y)\) are analytic functions without constant and linear terms defined in a certain neighborhood of the origin.

The aim of this paper consists on giving an upper bound for the number of algebraically independent Poincaré-Liapunov constants in a certain basis for planar polynomial differential systems. The final objective would be to find an upper bound for the number of functionally independent Poincaré-Liapunov constants for planar polynomial differential systems. If this upper bound existed, taking into account the works of Shi Songling, see [21], and under certain hypothesis about the Poincaré-Liapunov constants generators of the Dulac ideal, for instance the Dulac ideal to be radical, this upper bound would give an upper bound for the maximum number of small limit cycles which bifurcate from a nondegenerate singular point of focus type of a planar polynomial differential system.

We say that the analytic functions \(f_1, \ldots, f_m\) are functionally dependent in a neighborhood of the point \(p\), where \(f_i(p) = 0\) for \(i=1, \ldots, m\), if and only if there exist analytic functions \(g_1, \ldots, g_m\) in this neighborhood such that \(\sum_{i=1}^{m} f_i g_i \equiv 0\) and \(\sum |g_i|^2(p) \neq 0\). An ideal \(J\) of a ring \(A\) is radical if and only if \(\text{rad}(J) = J\), where \(\text{rad}(J) = \{ f \in A : f^n \in J \text{ for some } n > 0 \}\). The equivalent definition states that if \(f^n \in J\), then \(f \in J\).
Therefore, we consider the two–dimensional autonomous differential system

\[
\begin{align*}
\dot{x} &= \lambda x - y + X(x, y), \\
\dot{y} &= x + \lambda y + Y(x, y),
\end{align*}
\]

where \(X(x, y)\) and \(Y(x, y)\) are polynomials of degree \(m\), with \(m \geq 2\). Hence, the nonlinear terms can be expressed as 

\[
X(x, y) = \sum_{s=2}^{m} X_s(x, y) \quad \text{and} \quad Y(x, y) = \sum_{s=2}^{m} Y_s(x, y),
\]

where \(X_s(x, y)\) and \(Y_s(x, y)\) are homogeneous polynomials of degree \(s\), that is, 

\[
X_s(x, y) = \sum_{k=0}^{s} a_k x^k y^{s-k} \quad \text{and} \quad Y_s(x, y) = \sum_{k=0}^{s} b_k x^k y^{s-k}
\]

where \(a_k\) and \(b_k\) are arbitrary coefficients.

The analytic technique to set up the so-called nondegenerate centre problem was introduced by Poincaré [19]. More precisely, his solution consists in determining when a system of the form (1) has a local analytic first integral at the origin, and consequently a center at this point. Poincaré’s method consists on finding a formal power series of the form

\[
H(x, y) = \sum_{n=2}^{\infty} H_n(x, y),
\]

where \(H_2(x, y) = (x^2 + y^2)/2\), and for each \(n\), \(H_n(x, y)\) are homogeneous polynomials of degree \(n\), so that 

\[
H = \sum_{k=2}^{\infty} V_{2k}(x^2 + y^2)^k,
\]

where \(V_{2k}\) are called the Poincaré-Liapunov constants. For the polynomial system (2) the Poincaré-Liapunov constants are polynomials whose variables are the coefficients of system (2).

In order to solve the problem of the stability at the origin of system (1), it is sufficient to consider the sign of the first Poincaré-Liapunov constant different from zero. If it is positive we have asymptotic stability for negative times, and if it is negative we have asymptotic stability for positive times. If all Poincaré-Liapunov constants are zero, then the origin is stable for all times, but there is no asymptotic stability for any time, see for instance [2]. In this last case, we have a center at the origin, i.e. there is an open neighborhood of the origin where all orbits are periodic, except of course the origin. The origin is said to be a fine focus of order \(k\) if \(V_{2k+2}\) is the first non-zero Poincaré-Liapunov constant. In this case at most \(k\) limit cycles can bifurcate from this fine focus [5]; these limit cycles are called small-amplitude limit cycles. Therefore to obtain the maximum number of limit cycles which can bifurcate from the origin for a given system, one has to find the maximum possible order of a fine focus. It is known that this maximum number is three for quadratic system [4], see also [20] for the first part of the proof. Žoładek gave an algebraic proof of the Bautin Theorem [23]. An
analogous algebraic theorem holds for the space of vector fields the linear part of which are centers and the non-linear part are homogeneous polynomials of degree 3. The result says that the maximum number for these systems is five, see [24]. It has been shown recently, by the same author, that it is greater or equal than eleven for general cubic systems using Abelian integrals [25].

The proofs of these theorems depend essentially on the knowledge of all centers for the studied system, which makes possible to establish the structure of the coefficients in the Taylor expansion of the Poincaré map. Żoładek believes that probably a similar situation takes place for a fixed degree. Knowing sufficiently many cases of integrability we should be able to prove a corresponding algebraic theorem analog to the Bautin’s one, see [24]. Unfortunately, the knowledge of all centers is an open problem even for cubic systems. In this paper an upper bound for the number of algebraically independent Poincaré-Liapunov constants in a certain basis is given without the knowledge of all centers of the studied systems.

Let $E$ be some space of planar differential systems. We say that a nondegenerate singular point of focus type $p$ of system (1) has cyclicity $k$ with respect to the space $E$ if any perturbation in $E$ of this system (1) has at most $k$ limit cycles in a neighborhood of the point $p$ and $k$ is the minimal number with this property.

Shi Songling [22] proved that for polynomial systems we have uniqueness for the $V_{2k}$ in the following sense. Let $\mathbf{A}$ be the ring of real polynomials whose variables are the coefficients of the polynomial differential system. Given a set of Poincaré-Liapunov constants $V_1, V_2, \ldots, V_{i}$, let $\mathbf{J}_{k-1}$ be the ideal of $\mathbf{A}$ generated by $V_1, V_2, \ldots, V_{k-1}$. If $V'_1, V'_2, \ldots, V'_i$ is another set of Poincaré-Liapunov constants, then $V_k \equiv V'_k \pmod{(\mathbf{J}_{k-1})}$. As it has been said above the origin is a center if and only if all the Poincaré-Liapunov constants are zero. Let $\mathbf{J} = < V_1, V_2, \ldots >$ be the ideal of $\mathbf{A}$ generated by all the Poincaré-Liapunov constants. For polynomials systems, using the Hilbert’s basis theorem, $\mathbf{J}$ is finitely generated; i.e., there exist $B_1, B_2, \ldots, B_q$ in $\mathbf{J}$ such that $\mathbf{J} = < B_1, B_2, \ldots, B_q >$. Such a set of generators is called a basis of $\mathbf{J}$.

Notice that Hilbert’s basis theorem assures us the existence of a generators basis, but it does not provide us a constructive method to find it. The existent methods to solve this problem are based in the Buchberger’s algorithm to find a Gröbner basis, see [11]. Moreover, it is only applicable for very simple cases, see for instance [8, 9, 13, 14]. Therefore it is a computational problem of algebraic nature due to the appearance, already for simple systems, of massive Poincaré-Liapunov
constants that are polynomials with rational coefficients and efficient algorithms do not exist that allow to determine simple groups of generators. One of the main difficulties comes ultimately on the decomposition in prime numbers of a big integer number. Therefore the resolution of the computational problem goes to have efficient algorithms that work with big integers and in decomposition in primes numbers of big numbers, a classical problem in computational mathematics. On the other hand there are recursive methods for the determination of these Poincaré-Liapunov constants and the development of the algebraic manipulators has allowed to approach the calculation of the first constants, see for instance [15, 16].

Shi Songling also proved in [21] that for polynomials of degree \( m \), under certain hypothesis about the constants of Poincaré-Liapunov generators of the ideal, the maximum number of limit cycles is given by \( M(m) \), being \( M(m) \) the minimum number of the ideal generators. An open problem nowadays is the determination of \( M(m) \), or in its default an upper bound of it.

2. Some preliminary results

In the study of the center problem for systems of the form (2) we have used polar coordinates. In Lemma 1 we give the expression of system (2) in polar coordinates. In Proposition 2 we give the evaluation of the formal power series (3) for system (5) in these coordinates.

**Lemma 1.** In polar coordinates \( x = r \cos \varphi, \ y = r \sin \varphi \) we can write system (2) as

\[
\begin{align*}
\dot{r} &= \lambda r + \sum_{s=2}^{m} P_s(\varphi) r^s, \\
\dot{\varphi} &= 1 + \sum_{s=2}^{m} Q_s(\varphi) r^{s-1},
\end{align*}
\]

where \( P_s(\varphi) \) and \( Q_s(\varphi) \), are trigonometric polynomials of the form

\[
\begin{align*}
P_s(\varphi) &= R_{s+1} \cos \left( (s+1)\varphi + \varphi_{s+1} \right) + R_{s-1} \cos \left( (s-1)\varphi + \varphi_{s-1} \right) \\
&+ \cdots + \begin{cases} R_1 \cos (\varphi + \varphi_1) & \text{if } s \text{ is even;} \\
R_0 & \text{if } s \text{ is odd;}
\end{cases} \\
Q_s(\varphi) &= -R_{s+1} \sin \left( (s+1)\varphi + \varphi_{s+1} \right) + r_{s-1} \sin \left( (s-1)\varphi + \varphi_{s-1} \right) \\
&+ \cdots + \begin{cases} r_1 \sin (\varphi + \varphi_1) & \text{if } s \text{ is even;} \\
r_0 & \text{if } s \text{ is odd;}
\end{cases}
\end{align*}
\]

with \( R_j, r_j, \varphi_j \) and \( \bar{\varphi}_j \) arbitrary coefficients.

The formal power series (3) written in polar coordinates becomes

\[
H(r, \varphi) = \sum_{n=2}^{\infty} H_n(\varphi) r^n,
\]

with \( H_2(\varphi) = 1/2 \), and \( H_n(\varphi) \) a homogeneous trigonometric polynomial of degree \( n \), verifying \( H(r, \varphi) = \)
$$\lambda r^2 + \sum_{k=2}^{\infty} V_{2k} r^{2k}.$$ From the evaluation of $\dot{H}(r, \varphi)$ for system (2) the following result is established

$$\frac{dH_n(\varphi)}{d\varphi} + \lambda H_n(\varphi) + \sum_{k=1}^{n-2} (k + 1) H_{k+1}(\varphi) P_{n-k}(\varphi)$$

$$+ \sum_{k=1}^{n-2} \frac{dH_{k+1}(\varphi)}{d\varphi} Q_{n-k}(\varphi) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ V_n, & \text{if } n \text{ is even;} \end{cases}$$

with $n = 3, 4, \ldots$, and $H_2(\varphi) = 1/2$.

We will consider first, a system of the form (2) with $X(x, y) \equiv X_s(x, y)$ and $Y(x, y) \equiv Y_s(x, y)$, where $X_s(x, y)$ and $Y_s(x, y)$ are homogeneous polynomials of degree $s$, with $s \geq 2$. From now on, we call these systems homogeneous systems. In polar coordinates these systems are written as

$$(5) \dot{r} = \lambda r + P_s(\varphi)r^s, \quad \dot{\varphi} = 1 + Q_s(\varphi)r^{s-1},$$

where $P_s(\varphi)$ and $Q_s(\varphi)$, are trigonometric polynomials of the form of Lemma 1. The first Poincaré-Liapunov constant of system (5) is $V_2 = \lambda$. If we suppose $\lambda = 0$, the next proposition gives us a formally simple method for the computation of the Poincaré-Liapunov constants.

**Proposition 2.** In polar coordinates the formal power series (3) for the system (5) is

$$H(r, \varphi) = \sum_{m=0}^{\infty} \overline{H}_m(\varphi)r^{m(s-1)+2}$$

where $\overline{H}_m(\varphi)$, $m = 0, 1, \ldots$, are homogeneous trigonometric polynomials of degree $m(s-1) + 2$, satisfying the differential recurrence equations

$$\frac{d\overline{H}_{m+1}}{d\varphi} + (m(s - 1) + 2) \overline{H}_m(\varphi) P_s(\varphi) + \frac{d\overline{H}_m}{d\varphi} Q_s(\varphi)$$

$$= \begin{cases} 0, & \text{if } (m + 1)(s - 1) + 2 \text{ is odd;} \\ V_{(m+1)(s-1)+2}, & \text{if } (m + 1)(s - 1) + 2 \text{ is even;} \end{cases}$$

where $V_{(m+1)(s-1)+2}$, for $m = 0, 1, \ldots$, are the Poincaré-Liapunov constants.

Lemma 1 and Proposition 2 are proved in [7].

The second and the third Poincaré-Liapunov constants of system (5) are

$$V_{(s-1)+2} = -\frac{1}{2\pi} \int_0^{2\pi} P_s(\varphi)d\varphi = \begin{cases} -R_0, & \text{if } s \text{ is odd;} \\ 0, & \text{if } s \text{ is even;} \end{cases}$$

$$V_{2(s-1)+2} = \frac{1}{2\pi} \int_0^{2\pi} P_s(\varphi)Q_s(\varphi)d\varphi$$
\[
= \frac{1}{2} \left( \sum_{i=1}^{\lfloor s/2 \rfloor} R_{s-(2i-1)} r_{s-(2i-1)} \sin(\pi_{s-(2i-1)} - \varphi_{s-(2i-1)}) \right).
\]

We can write the homogeneous trigonometric polynomials \(P_s(\varphi)\) and \(Q_s(\varphi)\) of the form

\[
P_s(\varphi) = \pi_{s+1} \sin((s+1)\varphi) + \bar{b}_{s+1} \cos((s+1)\varphi) + \pi_{s-1} \sin((s-1)\varphi) + \cdots + \left\{ \frac{\bar{\pi}_1 \sin \varphi + \bar{b}_1 \cos \varphi}{\pi_0} \right\} \text{ if } s \text{ is even;} \\
Q_s(\varphi) = \tau_{s+1} \sin((s+1)\varphi) + \bar{d}_{s+1} \cos((s+1)\varphi) + \tau_{s-1} \sin((s-1)\varphi) + \cdots + \left\{ \frac{\tau_1 \sin \varphi + \bar{d}_1 \cos \varphi}{\tau_0} \right\} \text{ if } s \text{ is odd;}
\]

where there is a straightforward relation between the arbitrary coefficients \(\pi_j, \bar{b}_j, \tau_j\) and \(\bar{d}_j\) and the arbitrary coefficients \(R_j, r_j, \varphi_j\) and \(\bar{\varphi}_j\) which appear in Lemma 1.

In the same way we can write the homogeneous trigonometric polynomial \(H_m(\varphi)\) of degree \(m(s-1)+2\) of the form

\[
H_m(\varphi) = A_{m(s-1)+2}^m \sin((m(s-1)+2)\varphi) + B_{m(s-1)+2}^m \cos((m(s-1)+2)\varphi) + \cdots + \left\{ \frac{A_1^m \sin \varphi + B_1^m \cos \varphi}{A_0^m} \right\} \text{ if } s \text{ is even;} \\
\]

where \(A_k^m\) and \(B_k^m\) are homogenous polynomials of degree \(k\) in the variables \(\pi_j, \bar{b}_j, \tau_j\) and \(\bar{d}_j\) for \(k = 1, \ldots, m(s-1)+2\) if \(s\) is even and \(k = 0, \ldots, m(s-1)+2\) if \(s\) is odd.

3. Statement and Proof of the Main Results

**Proposition 3.** If \(\varpi\) is a polynomial which belongs to the Dulac ideal of the Poincaré-Liapunov constants, then this polynomial \(\varpi\) also belongs to the maximal ideal generated by the arbitrary coefficients \(\pi_j, \bar{b}_j, \tau_j\) and \(\bar{d}_j\) of the homogeneous trigonometric polynomials \(P_s(\varphi)\) and \(Q_s(\varphi)\) of system (5).

**Proof:** The proof follows easily taking into account the form of the differential recurrence equations of Lemma 1 and the way that appear the
terms of the Poincaré-Liapunov constants. In fact what happens is that
\[ \langle V_1, V_2, \ldots \rangle \subset \langle \bar{a}_j, \bar{b}_j, \bar{c}_j, \bar{d}_j \rangle. \] Moreover, it is easy to see that the maximal ideal generated by the arbitrary coefficient \( \langle a_j, b_j, c_j, d_j \rangle \) coincide with the maximal ideal generated by the arbitrary constants in cartesian coordinates \( \langle a_k, b_k \rangle \). Therefore we can write the following corollary.

**Corollary 4.** If \( \varpi \) is a polynomial which belongs to the Dulac ideal of the Poincaré-Liapunov constants, then this polynomial \( \varpi \) also belongs to the maximal ideal generated by the arbitrary coefficients in cartesian coordinates of system (5).

Given the set of Poincaré-Liapunov constants \( V_1, V_2, \ldots \), we can construct the set of Poincaré-Liapunov quantities \( V_1, V_2, \ldots \). To calculate this set, we reduce each \( V_k \) modulo \( \langle V_1, V_2, \ldots, V_{k-1} \rangle \) the ideal generated by the previous Poincaré-Liapunov constants \( V_1, V_2, \ldots, V_{k-1} \). Therefore, the set of Poincaré-Liapunov quantities \( V_1, V_2, \ldots \) consists of non-zero polynomials \( V_k \neq 0 \) for all \( k \), obtained by this procedure; the elements are \( V_k = V_k/\langle V_1, V_2, \ldots, V_{k-1} \rangle \), see for instance [5]. Evidently, if \( \varpi \) is a polynomial which belongs to the Dulac ideal of the Poincaré-Liapunov quantities, then this polynomial \( \varpi \) also belongs to the maximal ideal generated by the arbitrary coefficients in cartesian coordinates of system (5).

From the above results we know that any \( \varpi \) which belongs to the Dulac ideal of the Poincaré-Liapunov quantities also belongs to the maximal ideal generated by the arbitrary coefficients in cartesian coordinates of system (5). The number of arbitrary coefficients in cartesian coordinates of system (5) is \( 2(s + 1) \) because \( s + 1 \) is the number of arbitrary constants of a homogeneous polynomial of degree \( s \). Therefore, we can construct the following algebraic system of the first \( 2(s + 1) \) Poincaré-Liapunov quantities.

\[
\begin{align*}
V_1 &= m_{1,1} a_1 + m_{1,2} a_2 + \ldots + n_{1,s} b_s + n_{1,s+1} b_{s+1}, \\
V_2 &= m_{2,1} a_1 + m_{2,2} a_2 + \ldots + n_{2,s} b_s + n_{2,s+1} b_{s+1}, \\
& \quad \vdots \\
V_{2(s+1)} &= m_{2(s+1),1} a_1 + m_{2(s+1),2} a_2 + \ldots + n_{2(s+1),s} b_s + n_{2(s+1),s+1} b_{s+1},
\end{align*}
\]

where \( m_{i,j} \) and \( n_{i,j} \) are polynomials in the variables \( a_k \) and \( b_k \). We remark that the set \( \{ V_1, V_2, \ldots, V_{2(s+1)} \} \) is the product after an elimination procedure from \( V_1, V_2, \ldots \). The algebraic system (6) allows to express the variables \( a_k \) and \( b_k \) in function of the Poincaré-Liapunov
quantities $V_1, V_2, \ldots, V_{2(s+1)}$ if $\det A \neq 0$, where $\det A$ is the determinant of matrix $A$ associated to the algebraic system (6). Hence, we obtain the following result.

**Proposition 5.** Any $V_k$ for $k > 2(s+1)$ is always expressible in function of the set of the Poincaré-Liapunov quantities $V_1, \ldots, V_{2(s+1)}$ if $\det A \neq 0$, of the form

\[(7) \quad \det A \ V_k = m_{k,1} \ V_1 + m_{k,2} \ V_2 + \ldots + n_{k,s} \ V_{2s+1} + n_{k,s+1} \ V_{2(s+1)},\]

where $m_{k,j}$ and $n_{k,j}$ are polynomials in the variables $a_k$ and $b_k$ for $j = 1, \ldots, s+1$.

The columns of this matrix $A$ are the components of the set of the Poincaré-Liapunov quantities $V_1, V_2, \ldots, V_{2(s+1)}$ in the basis $\{a_k, b_k\}$. We call these vectors $v_1, v_2, \ldots, v_{2(s+1)}$, where $v_i$ is the vector associated to the Poincaré-Liapunov quantity $V_i$ in the basis $\{a_k, b_k\}$.

**Theorem 6.** This set of vectors $v_1, v_2, \ldots, v_{2(s+1)}$, which are the components of the set of the Poincaré-Liapunov quantities $V_1, V_2, \ldots, V_{2(s+1)}$ in the basis $\{a_k, b_k\}$, are algebraically dependent in a point $p$ of the center manifold $A^c_n$ of system (5).

**Proof:** Let $p$ be a point of the center manifold $A^c_n$ of system (5). Taking into account that $V_i(p) = 0$, for $i = 1, \ldots, 2(s+1)$. We have that the ordinary scalar product $\langle \overline{v}_i(p), \overline{v}(p) \rangle = 0$, where $\overline{v}$ is the vector $\overline{v} = (a_1, a_2, \ldots, a_{s+1}, b_1, b_2, \ldots, b_{s+1})$. The vectors $\overline{v}_1(p), \overline{v}_2(p), \ldots, \overline{v}_{2(s+1)}(p)$ are algebraically dependent because all these vectors are orthogonal with respect to the vector $v(p)$. Then, there exists a linear combination

\[(8) \quad c_1 \overline{v}_1(p) + c_2 \overline{v}_2(p) + \ldots + c_{2(s+1)} \overline{v}_{2(s+1)}(p) = 0 ,\]

where at least one $c_i$ is different from zero, with $c_j \in \mathbb{R}$ for $j = 1, \ldots, 2(s+1)$.

Taking an arbitrary $p$ belonging to a particular center stratus, $c_j$ are polynomials in the arbitrary coefficients which parametrize this center stratus. In case that total number of non-zero Poincaré-Liapunov quantities is less than $2(s+1)$ all the results stated in this work remain true. Therefore, there are at most $2s + 1$ algebraically independent in the set $\overline{v}_1(p), \overline{v}_2(p), \ldots, \overline{v}_{2(s+1)}(p)$ and taking into account that we can always reduce two parameters: one by scaling and one by rotation, we can write the following corollary.

**Corollary 7.** An upper bound for the number of algebraically independent Poincaré-Liapunov quantities in the basis $\{a_k, b_k\}$ in a point $p$ of the center manifold $A^c_n$ of system (5), is at most $2s - 1$. 
One consequence of Corollary 7 is that if we evaluate the determinant \( \det A \) of the algebraic system (6) in a point \( p \) of the center manifold \( \mathcal{A}_n^c \) of system (5) we always obtain \( \det A(p) = 0 \). This implies that \( \det A \in \langle V_1, V_2, \ldots \rangle \) the ideal generated by the Poincaré-Liapunov quantities \( V_1, V_2, \ldots \), and therefore equality (7) do not say anything about whether \( V_k \) belongs or not to the ideal \( \langle V_1, V_2, \ldots, V_{2(s+1)} \rangle \), for \( k > 2(s + 1) \) in a point \( p \) of the center manifold \( \mathcal{A}_n^c \) of system (5).

We have proved that for each point \( p \) of the center manifold \( \mathcal{A}_n^c \) of system (5) we can write the equality (8) and for and arbitrary point \( p \) if the \( \det A \not\equiv 0 \) we have equality (7), but we do not know if this fact implies the existence of an equality of the form \( g_1 V_1 + g_2 V_2 + \ldots + g_{2(s+1)} V_{2(s+1)} = 0 \) where at least one \( g_i \) is different from zero in each point \( p \in \mathcal{A}_n^c \), i.e., the existence of an upper bound for the number of the functionally independent Poincaré-Liapunov quantities. All the known results encourage the following conjecture.

**Conjecture 8.** An upper bound for the number of functionally independent Poincaré-Liapunov quantities is \( 2s - 1 \).

The following theorem gives the relationship between the cyclicity and the number of functionally independent Poincaré-Liapunov constants, and it is proved in [26].

**Theorem 9.** The cyclicity of a nondegenerate singular point of focus type of system (2) with respect to \( \mathcal{A}_n \) the space of planar polynomial differential systems of degree \( \leq n \) is greater or equal to the number of functionally independent generators of the Dulac ideal in a neighborhood of system (2). In particular, the cyclicity of a nondegenerate singular point of center type of system (2) from an irreducible component \( \Sigma \) of the center manifold \( \mathcal{A}_n^c \) is greater or equal to \( \text{codim} \Sigma - 1 \). If the Dulac ideal \( J \in \mathcal{A} \) is radical then the cyclicity of a nondegenerate singular point of focus type of system (2) is equal to the number of functionally independent generators of the Dulac ideal near system (2).

Taking into account Theorem 9 and if the Conjecture 8 is true, when the Dulac ideal is radical the cyclicity is less or equal to \( 2s - 1 \). This consequence can be tested for the case \( s = 2 \) and for the case \( s = 3 \) because in these cases the Dulac ideal is radical. In both cases the number of small limit cycles is 3 and 5 respectively, see for instance [4] and [24]. The conjectured upper bound for these cases gives the result of 3 and 5 respectively. These values are realizable, see for instance [4] and [5] respectively. The center variety is connected by the arbitrary unessential coefficients which contribute to the codimension in each case.
The same results can be obtained for the non homogeneous system (4). In this case the number of arbitrary coefficients in cartesian coordinates is \(2 \sum_{i=2}^{m}(s+1) = (m+4)(m-1) = m^2 + 3m - 4\) and therefore, the number of algebraically independent Poincaré-Liapunov quantities is \(m^2 + 3m - 5\). In the same way of the homogeneous case, we can always reduce two parameters: one by scaling and one by rotation and establish a similar result to Theorem 6 for the non homogeneous case and a similar conjecture as Conjecture 8. Taking into account these facts, we would obtain that an upper bound for the functionally independent Poincaré-Liapunov quantities would be \(m^2 + 3m - 7\). This value coincides with the value conjectured by Žołdek in [25]. This author also established the following conjecture.

**Conjecture 10.** Each ideal \(J \in A\) of the Poincaré-Liapunov constants of the planar vector fields of degree \(\leq n\) is radical.

**References**


[4] N.N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Mat. Sb. 30 (72) (1952), 181–196; Amer. Math. Soc. Transl. 100 (1954) 397–413.


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