On the Weak Distance-regularity of Moore-type Digraphs*

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WEAK DISTANCE-REGULARITY MOORE DIGRAPHS

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Abstract

We prove that Moore digraphs, and some other classes of extremal digraphs, are weakly distance-regular in the sense that there is an invariance of the number of walks between vertices at a given distance. As weakly distance-regular digraphs, we then compute their complete spectrum from a ‘small’ intersection matrix. This is a very useful tool for deriving some results about their existence and/or their structural properties. For instance, we present here an alternative and unified proof of the existence results on Moore digraphs, Moore bipartite digraphs and, more generally, Moore generalized $p$-cycles. In addition, we show that the line digraph structure appears as a characteristic property of any Moore generalized $p$-cycle of diameter $D \geq 2p$. 
1 Introduction

Let us first introduce some basic notation (see [11]) and discuss the relevant background. Let \( \Gamma = (V,E) \) be a (strongly) connected digraph with order \( N := |V| \) and diameter \( D \). We will denote by \( \text{dist}(u,v) \) the distance from vertex \( u \) to vertex \( v \). Notice that \( \text{dist}(u,v) \) may not be equal to \( \text{dist}(v,u) \) since we are considering directed graphs. A graph will be identified with the symmetric digraph obtained by replacing each undirected edge by a pair of opposite arcs, and vice versa. For any fixed integer \( 0 \leq k \leq D \), we will denote by \( \Gamma^+_{k}(v) \) (respectively, \( \Gamma^-_{k}(v) \)) the set of vertices at distance \( k \) from \( v \) (respectively, the set of vertices from which \( v \) is at distance \( k \)). Thus, the \textit{out-degree} and \textit{in-degree} of \( v \) are \( \delta^+(v) := |\Gamma^+_{1}(v)| \) and \( \delta^-(v) := |\Gamma^-_{1}(v)| \); and the digraph \( \Gamma \) is \textit{\(-\Delta\)-regular} if \( \delta^+(v) = \delta^-(v) = \Delta \) for every \( v \in V \).

Notice that if \( \Gamma \) is a symmetric digraph, then \( \Gamma^+_{k}(v) = \Gamma^-_{k}(v) \) (:= \( \Gamma_{k}(v) \), for short). We recall that the adjacency matrix \( A = (a_{uv}) \) of the digraph \( \Gamma \) is the \( N \times N \) matrix indexed by the vertices of \( \Gamma \), with entries \( a_{uv} = 1 \) if \( u \) is adjacent to \( v \), and \( a_{uv} = 0 \) otherwise. The distance-\( k \) matrix \( A_k \) of a digraph \( \Gamma \) with diameter \( D \), where \( 0 \leq k \leq D \), is defined by

\[
(A_k)_{uv} := \begin{cases} 
1 & \text{if } \text{dist}(u,v) = k, \\
0 & \text{otherwise.}
\end{cases}
\]  

In particular, \( A_0 = I \) and \( A_1 = A \) (we assume that \( \Gamma \) contains neither loops nor multiple edges).

Spectrum

The spectrum of a digraph \( \Gamma \), denoted by \( \text{sp} \Gamma \), consists of the eigenvalues of its adjacency matrix \( A \) together with their (algebraic) multiplicities,

\[
\text{sp} \Gamma := \text{sp} A = \begin{pmatrix} 
\lambda_0 & \lambda_1 & \ldots & \lambda_d \\
m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_d)
\end{pmatrix}.
\]

The knowledge of the spectrum of a (di)graph is relevant for estimating some of its structural properties, which provide information on the topological and communication properties of the corresponding network. Among these properties we have, for instance, edge-expansion and node-expansion, bisection width, diameter, maximum cut, connectivity, and partitions (see [12, 36]). Besides, the estimation of connectivity and expansion parameters of graphs are, in general, very hard to obtain by other methods. Efficient graph partition algorithms have been constructed based on the eigenvalues and eigenvectors (see [28, 29]). The spectrum of a graph has also been considered to provide load balancing algorithms (see [6]).

Distance-regularity

Distance-regular graphs were introduced by Biggs [8], by changing a symmetry-type requirement, that of distance-transitivity, by a regularity-type condition concerning the cardinality of some vertex subsets. To be more precise, recall that a graph \( \Gamma \) with diameter \( D \) is \textit{distance-regular} if, for any pair of vertices \( u,v \in V \) such that \( \text{dist}(u,v) = k \), \( 0 \leq k \leq D \), the numbers

\[
p^k_{\Gamma_{1}}(u,v) := |\Gamma_{1}(u) \cap \Gamma_{1}(v)|
\]  


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do not depend on the chosen vertices $u$ and $v$, but only on their distance $k$; in which case we just write $p^{k}_{i1}(u,v) = p^{k}_{i1}$ for some constants $p^{k}_{i1}$ called the intersection numbers (note that $p^{k}_{i1} = 0$ when $|i - k| > 1$).

Damerell [17] adopted the strongest definition of distance regularity for digraphs. Thus, he defined a digraph $\Gamma$ with diameter $D$ to be (strongly) distance-regular if, for any pair of vertices $u, v \in V$ such that $\text{dist}(u,v) = k$, $0 \leq k \leq D$, the numbers

$$s^{k}_{i1}(u,v) := |\Gamma^{+}_{i}(u) \cap \Gamma^{+}_{1}(v)|$$

(3)
do not depend on the chosen vertices $u$ and $v$, but only on their distance $k$; in which case they are denoted by $s^{k}_{i1}$ (note that $s^{k}_{i1} = 0$ when $i > k + 1$). Damerell [17] proved that every distance-regular digraph $\Gamma$ with girth $g$, say, is stable; that is,

$$\text{dist}(u,v) + \text{dist}(v,u) = g, \text{ if } 0 < \text{dist}(u,v) < g.$$  

(4)

Thus, a stable digraph of girth two is a symmetric digraph and, consequently, can be identified with a graph. Therefore, the case $g = 2$ corresponds to the distance-regular graphs. Damerell also proved that the set of distance-matrices $\{A^{k}\}_{k=0}^{D}$ of a distance-regular digraph $\Gamma$ define a commutative association scheme on the vertices of $\Gamma$ (see [17]). Then, using Higman’s results on such configurations (see [30]), he showed that the standard theorems about distance-regular graphs (see e.g. [7, 26]) hold also for digraphs.

If we change $\Gamma^{+}_{i}(v)$ by $\Gamma^{-}_{i}(v)$ in the definition of distance-regularity (3), we get some new parameters $p^{k}_{i1}$. But now, as we showed in [14], the stability property (4) does not necessarily hold, and a class of digraphs with less structure appears. In that paper, we considered such digraphs, referred to as ‘weakly distance-regular’, and studied some of their properties. In particular, we proved that every distance-regular digraph is also weakly distance-regular, so justifying the name, and the defining parameters, $s^{k}_{i1}$ and $p^{k}_{i1}$, satisfy that $N_{k}p^{k}_{i1} = N_{i}s^{k}_{i1}$, where $N_{k}$ denotes the number of (ordered) vertex pairs $u,v$ such that $\text{dist}(u,v) = k$. Furthermore, we showed that distance-regular digraphs are precisely those weakly distance-regular digraphs which are stable.

Weakly distance-regularity

The concept of a weakly distance-regular digraph was introduced by the authors [14], through the invariance of the number of walks between vertices at a given distance. (This approach has already been used to characterize distance-regular graphs, see e.g. [38].) Formally, a digraph $\Gamma$ of diameter $D$ is weakly distance-regular if, for each non-negative integer $l \leq D$, the number $a^{(l)}_{uv}$ of walks of length $l$ from vertex $u$ to vertex $v$ only depends on their distance $\text{dist}(u,v) = k$, for any $l = 0, 1, \ldots, D$.

From the definition of $\Gamma$ as a weakly distance-regular digraph, and taking into account that $a^{(l)}_{uv} = (A^{l})_{uv}$, it follows that each matrix power $A^{l}$, $0 \leq l \leq D$, can be expressed as a linear combination of the distance matrices $A_{0}, A_{1}, \ldots, A_{D}$. As a consequence, these matrices belong to the adjacency algebra of $\Gamma$, which is defined by

$$\mathcal{A}(\Gamma) := \{p(A) : p \in \mathbb{C}[x]\}.$$  

This leads to some characterizations of weakly distance-regular digraphs (see [14]) which are analogous to those applying for distance-regular graphs (for a survey of the latter, see e.g. [19]).
Theorem 1.1. Let \( \Gamma \) be a connected digraph of diameter \( D \). Then, \( \Gamma \) is a weakly distance-regular digraph if and only if any of the following statements hold:

(a) The distance matrix \( A_k \) is a polynomial of degree \( k \) in the adjacency matrix \( A \); that is, \( A_k = p_k(A) \), for each \( k = 0, 1, \ldots, D \), where \( p_k \in \mathbb{Q}[x] \).

(b) The set of distance matrices \( \{ A_k \}_{k=0}^D \) is a basis of the adjacency algebra \( A(\Gamma) \).

(c) For any two vertices \( u, v \in V(\Gamma) \) at distance \( d(u, v) = k \), the numbers

\[
p^k_{ij}(u, v) := |\Gamma^+_i(u) \cap \Gamma^-_j(v)|
\]

do not depend on the vertices \( u \) and \( v \), but only on their distance \( k \); in which case they are denoted by \( p^k_{ij} \) (note that \( p^k_{ij} = 0 \) when \( k > i + j \)).

Some examples of weakly distance-regular digraphs, such as the butterfly and the cycle prefix digraph which are interesting for their applications, are analyzed in [14].

If \( \Gamma \) is a weakly distance-regular digraph, then \( \sum_{k=0}^D p_k(A) = \sum_{k=0}^D A_k = J \), where \( J \) denotes the all-1 matrix. As a consequence, \( \Gamma \) is a regular digraph of degree \( \Delta \), say. The polynomials \( p_k \) such that \( A_k = p_k(A) \), \( 0 \leq k \leq D \), are referred to as the distance polynomials of \( \Gamma \) and the numbers \( p^k_{ij} = |\Gamma^+_i(u) \cap \Gamma^-_j(v)| \), where \( d(u, v) = k \), are also known as the intersection numbers of \( \Gamma \). Since

\[
A_i A_j = \sum_{k=0}^{\min(i+j,D)} p^k_{ij} A_k,
\]

the distance polynomials of \( \Gamma \) satisfy the recurrence relation (take \( j = 1 \))

\[
p_i x = \sum_{k=0}^{i+1} p^k_{i1} p_k \quad (0 \leq i \leq D - 1).
\]

As happens in the case of distance-regular graphs, the spectrum of a weakly distance-regular digraph can be retrieved from the information given by some different ‘small’ matrices characterizing it, such as the intersection matrix or the multiplicity matrix (see [14]). The intersection or recurrence matrix of a weakly distance-regular digraph \( \Gamma \) is the \((D+1)\times(D+1)\) matrix \( B \) whose entries are the intersection numbers \( p^k_{ij} \); that is \((B)_{ik} = p^k_{ik} \). In particular, all column sums of \( B \) equal to the degree \( \Delta \) of \( \Gamma \) since \( \sum_{k=0}^D p^k_{ij}(u, v) = |\Gamma^+_i(v)| \). We point out that this definition of \( B \) coincides with the given in [14], \((B)_{ik} = p^k_{1i} \), since \( A_i A_i = A_i A_1 \) \((A_i \in A(\Gamma), i = 0, 1, \ldots, D, \) and \( A(\Gamma) \) is a commutative algebra). We also notice that, while the intersection matrix \( B \) of a weakly distance-regular digraph is a lower Hessenberg matrix \((p^k_{1i} = 0 \text{ if } k > i + 1)\), in the undirected case \( B \) is tridiagonal. The following result, proved in [14], shows how to compute the spectrum of a weakly distance-regular digraph given its intersection matrix. We shall say that a vector \( u \) is standard if its first component is \((u)_0 = 1 \).

Proposition 1.2. Let \( \Gamma \) be a weakly distance-regular digraph of degree \( \Delta \), diameter \( D \), order \( N \), and distance polynomials \( \{ p_k \}_{k=0}^D \). Let \( A \) and \( B \) be, respectively, its adjacency and intersection matrices. Then, the following statements hold:
(i) The minimum polynomials of \( A \) and \( B \) coincide with the characteristic polynomial of \( B \) which is
\[
\det(xI - B) = \frac{1}{\alpha_D^D}(x - \Delta) \sum_{k=0}^{D} p_k,
\]
where \( \alpha_D^D \) is the leading coefficient of \( p_D \).

(ii) If \( \lambda_i \) is an eigenvalue of \( B \), then \( v_i = (p_0(\lambda_i), p_1(\lambda_i), \ldots, p_D(\lambda_i))^\top \) is a (right) standard eigenvector of \( B \).

(iii) If each eigenvalue \( \lambda_i \) of \( B \) is simple, then \( \lambda_i \) has unique left and right standard eigenvectors, \( u_i^\top \) and \( v_i \), and the multiplicity of \( \lambda_i \), as an eigenvalue of \( \Gamma \), is
\[
m(\lambda_i) = \frac{N}{u_i^\top v_i} \quad (0 \leq i \leq D). \tag{8}
\]

We point out that the adjacency matrix \( A \) of a distance-regular digraph \( \Gamma \) is normal, that is, \( AA^* = A^*A \), where \( A^* \) denotes the transpose of its conjugate, since \( A^\top = A_{g-1} = p_{g-1}(A) \). This is equivalent to say that \( A \) diagonalizes by means of a unitary matrix (see e.g. [34]).

Moore graphs and digraphs

It is well known that the order \( N \) of a connected graph (respectively, digraph) \( \Gamma \) with maximum degree (respectively, out-degree) \( \Delta \) and diameter \( D \) cannot be greater than the corresponding Moore bound,
\[
N \leq \begin{cases} M_G(\Delta, D) := 1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1} & \text{if } \Gamma \text{ is a graph,} \\ M_D(\Delta, D) := 1 + \Delta + \Delta^2 + \cdots + \Delta^D & \text{if } \Gamma \text{ is a digraph,} \end{cases}
\]
since the number of vertices of \( \Gamma \) at distance \( k \leq D \) from a given vertex is at most \( \Delta(\Delta - 1)^{k-1} \) or \( \Delta^k \), depending whether \( \Gamma \) is undirected or not, respectively. In the extremal case, \( N = M_G(\Delta, D) \) or \( N = M_D(\Delta, D) \), we have a Moore graph or a Moore digraph, respectively. Notice that in a Moore graph, if edges joining vertices at distance \( D \) from a distinguished vertex are deleted, the residual graph is a ‘hierarchy’ and the same ‘structure’ results from distinguishing any vertex (see [33]). This makes a Moore graph to be distance-regular in the intuitive sense that, when we ‘hang’ the graph from a given vertex, we observe that vertices in the same ‘layer’, or distance from the ‘root’, are ‘neighbourhood-indistinguishable’ from each other, and the whole ‘configuration’ does not depend on the chosen vertex. More precisely, a Moore graph \( \Gamma \) is distance-regular with intersection matrix
\[
B = \begin{pmatrix} 0 & 1 & & & \\ \Delta & 0 & 1 & & \\ & \Delta - 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \Delta - 1 & 1 \end{pmatrix},
\]
(see [7]). The question of the existence of (nontrivial) Moore graphs of degree \( \Delta > 2 \) and diameter \( D = 2, 3 \) was ‘almost’ settled by Hoffman and Singleton [33], by showing that there
is only one such graph for \( D = 2 \) and \( \Delta = 3, 7 \) and no other Moore graphs with the ‘possible’ exception of a graph with degree \( \Delta = 57 \) and diameter \( D = 2 \). To prove the nonexistence result they used spectral techniques. More precisely, they worked with some graph invariants given in terms of the eigenvalues of a Moore graph whose adjacency matrix \( A \) must satisfy the equation

\[
F_D(A) = J,
\]

where \( F_k = xF_{k-1} - (\Delta - 1)F_{k-2} \), if \( k \geq 2 \), and \( F_0 = 1 \), \( F_1 = x + 1 \). (9)

The nonexistence of Moore graphs of diameter \( D > 3 \) (and degree \( \Delta > 2 \)) was independently proved by Bannai and Ito [1] and Damerell [16], who derived the spectrum of a Moore graph \( \Gamma \) working with its intersection matrix \( B \), in the light of the theory of distance-regularity.

Nevertheless, the classification of Moore digraphs, first given by Plesnık and Znám [37], was done by directly working with the corresponding matrix equation, which is

\[
I + A + \cdots + A^D = J
\]

(10)

(see also [10, 15] for shorter proofs).

So, a different approach has been used to solve the Moore graph existence problem for the undirected and directed cases. One reason could be that Eq. (10) is much simpler than (9), since in the directed case we do not have to care about discounting walks that use the same edge twice (forward and backward). Another reason may be that the theory of distance-regular digraphs cannot be applied to this problem since nontrivial Moore digraphs are not (strongly) distance-regular.

We point out that, from (9) and (10), \( F_D \) [respectively, \( 1 + x + \cdots + x^D \)] is the \textit{Hoffman polynomial} (see [31, 32]) of a Moore graph \( \Gamma \) of diameter \( D \) [respectively, Moore digraph], since it is the least degree polynomial satisfying \( H(A) = J \). Moreover, \( F_D \) coincides with the sum of the distance polynomials of \( \Gamma \), which can also be defined in the directed case as we will see.

In this paper, we prove that Moore digraphs, and some other classes of extremal digraphs, are weakly distance-regular. This allow us to give an alternative and unified proof of the existence of Moore digraphs, Moore bipartite digraphs and Moore generalized \( p \)-cycles (see Sections 2 and 3). Besides, we consider a certain subclass of ‘almost’ Moore digraphs which are also weakly distance-regular (see Section 4). As another spectral application, we show that the line digraph structure appears as a characteristic property of several Moore-type digraphs (see Sections 3 and 4).

### 2 Moore digraphs revisited

In this section, we show that a Moore digraph of diameter \( D \) is the simplest example of a weakly distance-regular digraph, in the sense that its distance polynomials constitute the canonical basis of \( C_D[x] \).

**Proposition 2.1.** A Moore digraph \( \Gamma \) with maximum out-degree \( \Delta \) and diameter \( D \) is a weakly distance-regular digraph with distance polynomials \( p_k = x^k \), \( k = 0, 1, \ldots, D \). Moreover, \( \Gamma \) is distance-regular if and only if \( \Delta = 1 \) or \( D = 1 \), which corresponds to the cycle digraph \( C_{D+1} \) and to the complete digraph \( K_{\Delta+1} \), respectively.
Proof. A Moore digraph $\Gamma$ of diameter $D$ becomes a rooted tree at vertex $u$ if the arcs incident from vertices in $\Gamma_D^+ (u)$ are removed. That is, for each vertex $v \in V(\Gamma)$ there is exactly one $u \to v$ walk of length $k \leq D$ ($k = \text{dist}(u,v)$). This means that the distance-$k$ matrix $A_k$ of $\Gamma$ is just $A_k = A^k$, where $A$ is the adjacency matrix of $\Gamma$. Therefore, $\Gamma$ is a weakly distance-regular digraph with distance polynomials $p_k = x^k$, $k = 0, 1, \ldots, D$. In particular, $\Gamma$ is $\Delta$-regular.

Now, let us assume that a Moore digraph $\Gamma$ of degree $\Delta$ and diameter $D$ is distance-regular. Notice first that the girth of $\Gamma$ is $g = D + 1$. Then, from the stability property (4), $A_D = A_D^\top$ and, consequently, $A^D = A_D = A^\top$. Therefore, $A^D J = \Delta^D J = A^\top J = \Delta J$, since $\Gamma$ is $\Delta$-regular. Hence, $\Delta^D = \Delta$, which implies that $\Delta = 1$ or $D = 1$. In such trivial cases, $\Gamma$ is clearly distance-regular since it is either the cycle digraph of order $D + 1$ or the complete digraph of order $\Delta + 1$, respectively. □

Now we compute the spectrum of a Moore digraph in the light of the theory of weak distance-regularity.

**Theorem 2.2.** The spectrum of a Moore digraph $\Gamma$ of degree $\Delta$ and diameter $D$ is

$$\text{sp} \Gamma = \left( \begin{array}{cccccc}
\Delta & \xi & \xi^2 & \cdots & \xi^D \\
1 & m(\xi) & m(\xi^2) & \cdots & m(\xi^D) 
\end{array} \right),$$

where $\xi$ is a primitive $(D + 1)$th root of unity, say $\xi = e^{2\pi i/(D+1)}$, and

$$m(\xi^k) = N \frac{\Delta}{\xi^k - \Delta}, \quad k = 1, 2, \ldots, D,$$

where $N = 1 + \Delta + \cdots + \Delta^D$.

**Proof.** Since a Moore digraph $\Gamma$ of degree $\Delta$ and diameter $D$ is weakly distance-regular with distance polynomials $p_k = x^k$, $k = 0, 1, \ldots, D$, its intersection matrix $B = (p_{i1}^k)$ is

$$B = \begin{pmatrix}
0 & 1 & & & \\
0 & 0 & 1 & & \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
0 & \Delta & \Delta - 1 & \Delta - 1 & 1 & 0 & 1 \\
\Delta & \Delta - 1 & \Delta - 1 & \Delta - 1 & 1 & 0 & 1 \\
\end{pmatrix},$$

since $p_i x = \sum_{k=0}^{i+1} p_{i1}^k p_k = p_{i+1}$, which means that $p_{i1}^{i+1} = 1$ and $p_{i1}^k = 0$, otherwise, for each $i = 0, 1, \ldots, D - 1$. Taking into account that $p_{D1}^k = \Delta - \sum_{i=0}^{D-1} p_{i1}^k$, for each $k = 0, 1, \ldots, D$, the last row of $B$ is derived. Then, from Proposition 1.2, the minimum polynomial of the adjacency matrix $A$ of $\Gamma$ coincides with the characteristic polynomial of $B$, which is

$$\det(x I - B) = (x - \Delta) \sum_{k=0}^{D} p_k = (x - \Delta)(1 + x + \cdots + x^D) = (x - \Delta) \frac{x^{D+1} - 1}{x - 1} = (x - \Delta) \prod_{k=1}^{D} (x - \xi^k),$$

8
where \( \xi \) is a primitive \((D + 1)\)th root of unity, say \( \xi = e^{2\pi i/(D+1)} \). Therefore, since all the eigenvalues of \( B \) are simple, we can apply (8) to obtain their multiplicities as eigenvalues of \( \Gamma \). Thus, the multiplicity \( m(\xi) \) of \( \xi \) can be computed as

\[
m(\xi) = \frac{N}{u^\top_\xi v_{\xi}},
\]

where \( N \) is the order of \( \Gamma \) and \( v_{\xi} = (1, \xi, \ldots, \xi^D)^\top \), \( u^\top_\xi = (1, x_1, \ldots, x_D) \) represent the only right and left standard eigenvectors of \( B \) corresponding to \( \xi \), respectively. The condition \( u^\top_\xi B = \xi u^\top_\xi \) gives

\[
x_k = \frac{1}{\xi} x_{k-1} + \frac{\Delta - 1}{\Delta}, \quad k = 1, 2, \ldots, D \quad \text{and} \quad x_0 := 1,
\]

whence

\[
x_k = \frac{1}{\xi^k} + \frac{\Delta - 1}{\Delta} \frac{\xi^k - 1}{\xi - 1}, \quad k = 0, 1, \ldots, D.
\]

Then, taking into account that \( \xi^{D+1} = 1 \), we have

\[
u^\top_\xi v_{\xi} = \sum_{k=0}^{D} x_k \xi^k = \sum_{k=0}^{D} \left(1 + \frac{\Delta - 1}{\Delta} \frac{\xi}{\xi - 1} (\xi^k - 1)\right) = \frac{D + 1}{\Delta} \frac{\xi - \Delta}{\xi - 1}.\quad (11)
\]

Since in the previous computation we have only used that \( \xi \) is a \((D + 1)\)th root of unity, \( \xi \neq 1 \), we can extend (11) to any other root \( \xi^k \), \( k = 1, 2, \ldots, D \). Hence, the multiplicity of \( \xi^k \) is

\[
m(\xi^k) = \frac{N}{u^\top_\xi v_{\xi^k}} = \frac{\Delta}{D + 1} \frac{\xi^k - 1}{\xi^k - \Delta}.
\]

The multiplicity of the degree \( \Delta \) of \( \Gamma \) is 1 since \( JB = \Delta J \) and, consequently, \( u^\top_\Delta = (1, 1, \ldots, 1) \) and \( u^\top_\Delta v_{\Delta} = N \). Such a multiplicity can also be derived from Perron-Frobenius Theorem (see [34]).

Thus, for instance, the spectrum of a Moore digraph \( \Gamma \) of degree \( \Delta = 1 \) and diameter \( D \) is

\[
\text{sp } \Gamma = \begin{pmatrix} 1 & \xi & \cdots & \xi^D \\ 1 & 1 & \cdots & 1 \end{pmatrix},
\]

which corresponds to the cycle digraph of order \( D + 1 \). In the case \( D = 1 \), the spectrum of \( \Gamma \) is

\[
\text{sp } \Gamma = \begin{pmatrix} \Delta & -1 \\ 1 & \Delta \end{pmatrix},
\]

since \( \xi = -1 \) and \( m(\xi) = \Delta \), which corresponds to the complete digraph of order \( \Delta + 1 \).

**Corollary 2.3** ([37, 10, 15]). Moore digraphs of degree \( \Delta > 1 \) and diameter \( D > 1 \) do not exist.

**Proof.** Let \( \Gamma \) be a Moore digraph of degree \( \Delta \) and diameter \( D \). From Theorem 2.2, we know that \( \xi = e^{2\pi i/(D+1)} \) is an eigenvalue of \( \Gamma \) with multiplicity \( m(\xi) = \frac{\Delta}{D + 1} \frac{\xi - 1}{\xi - \Delta} \). Therefore, \( \frac{\xi - 1}{\xi - \Delta} \) is rational, since \( m(\xi) \) is a positive integer, which implies that \( \xi = e^{2\pi i/(D+1)} \) is rational, if \( \Delta > 1 \). But, this is so if and only if \( D = 1 \). \( \square \)
3 Moore generalized $p$-cycles

A generalized $p$-cycle is a digraph $\Gamma$ such that its set of vertices can be partitioned in $p \geq 1$ parts, $V(\Gamma) = \bigcup_{\alpha \in \mathbb{Z}_p} V_{\alpha}$, in such a way that the vertices in the partite set $V_{\alpha}$ are only adjacent to vertices in $V_{\alpha+1}$, where the sum is in $\mathbb{Z}_p$. Notice that bipartite digraphs are generalized $p$-cycles with $p = 2$ and all digraphs are included in the case $p = 1$.

The diameter $D$ of a strongly connected generalized $p$-cycle $\Gamma$ is the minimum integer such that for any vertex $v$, all vertices of one of the partite sets of $V(\Gamma)$ are at distance at most $D - (p - 1)$ from $v$. As a consequence, and taking also into account that the lengths of all walks from one partite set to another are congruent modulo $p$, the order $N$ of a generalized $p$-cycle with maximum out-degree $\Delta$ and diameter $D$ is bounded above by the Moore-like bound

$$N \leq M_{GC}(\Delta, D) := p^r(1 + \Delta^p + \cdots + \Delta^{pq}),$$

where $D - (p - 1) = pq + r$ and $0 \leq r \leq p - 1$; see [27]. Notice that $M_{GC}(\Delta, D)$ coincides with the (general) Moore bound $M_{BD}(\Delta, D)$, if $p = 1$, and in the case $p = 2$ is equal to the known Moore-like bound for bipartite digraphs (see [21])

$$M_{BD}(\Delta, D) := \begin{cases} 2(1 + \Delta^2 + \cdots + \Delta^{2q}) & \text{if } D = 2q + 1, \\ 2\Delta(1 + \Delta^2 + \cdots + \Delta^{2q-2}) & \text{if } D = 2q. \end{cases}$$

A generalized $p$-cycle $\Gamma = (V, E)$, $V = \bigcup_{\alpha \in \mathbb{Z}_p} V_{\alpha}$, with maximum out-degree $\Delta$, diameter $D$ and order $N = M_{GC}(\Delta, D)$ is called a Moore generalized $p$-cycle (in the case $p = 2$ it is referred to as a Moore bipartite digraph). In fact, such a digraph must be out-regular with out-degree $\Delta$; that is, $\delta^+(v) = \Delta$ for any $v \in V(\Gamma)$. The problem of the Moore generalized $p$-cycles existence was solved by Gómez et al. [27], as a natural extension of the techniques used by Fiol and Yebra [21] to deal with the bipartite case. Thus, it is known that the bound $M_{GC}(\Delta, D)$ is only attained when $p \leq D \leq 3p - 2$, if $\Delta > 1$.

In this section, we show that a Moore generalized $p$-cycle is also weakly distance-regular. Then, using the theory of weak distance-regularity, we present another proof of the nonexistence of Moore generalized $p$-cycles of degree $\Delta > 1$ and diameter $D > 3p - 2$. Besides, we discuss when a Moore generalized $p$-cycle is a line digraph.

**Proposition 3.1.** A Moore generalized $p$-cycle $\Gamma$ with out-degree $\Delta$ and diameter $D$, where $D - (p - 1) = pq + r$ and $0 \leq r \leq p - 1$, is weakly distance-regular and its distance polynomials are

$$p_k = x^k, \quad k = 0, 1, \ldots, D - (p - 1),$$

$$\begin{align*}
p_{pq+r+j} &= \begin{cases} \frac{1}{\Delta^j}(x^{pq+r+j} - (\Delta^j - 1)(x^{r+j} + x^{p+r+j} + \cdots + x^{p(q-1)+r+j})) & \text{if } r + j \leq p - 1, \\ \frac{1}{\Delta^j}(x^{pq+r+j} - (\Delta^j - 1)(x^{r+j} + x^{p+r+j} + \cdots + x^{p(q-1)+r+j})) - x^{r+j-p} & \text{if } r + j \geq p, \end{cases}
\end{align*}$$

for each $j = 1, 2, \ldots, p - 1$. Moreover, $\Gamma$ is distance-regular if and only if $\Delta = 1$ or $D = p$, which corresponds to the cycle digraph $C_{p(q+1)}$ and to the complete generalized $p$-cycle $K^{GC}_{\Delta_1,\ldots,\Delta_p}$, respectively.
correspond to the cycle digraph \( C \). Hence, \( \Gamma \) cannot be distance-regular unless \( \Delta = 1 \) (two vertices, respectively.

As we will see, the intersection matrix of a Moore generalized \( p \)-cycle of diameter \( D \) has the eigenvalue 0 that is not simple, if \( D - (p - 1) \equiv 0 \pmod{p} \). This means that, for any vertex \( v \in V(\Gamma) \) at distance \( \leq D - (p - 1) \) from \( u \), the number of \( u \to v \) walks of length \( k \), \( k = 0, 1, \ldots, D - (p - 1) \), is \( a^{(k)}_{uv} = 1 \), if \( \text{dist}(u, v) = k \), and \( a^{(k)}_{uv} = 0 \), otherwise. So, the distance \( k \)-matrix of \( \Gamma \) is

\[
A_k = A^k, \quad k = 0, 1, \ldots, D - (p - 1).
\]

Then, taking into account that \( \Gamma \) has out-degree \( \Delta \),

\[
a^{(pq+r+j)}_{uv} = \begin{cases} 
\Delta^j & \text{if } \text{dist}(u, v) = pq + r + j \text{ or } \text{dist}(u, v) = r + j - p, \\
\Delta^j - 1 & \text{if } \text{dist}(u, v) \equiv r + j \pmod{p} \text{ and } \text{dist}(u, v) \neq pq + r + j, r + j - p, \\
0 & \text{otherwise},
\end{cases}
\]

for each \( j = 1, 2, \ldots, p - 1 \). In matrix terms,

\[
A^{pq+r+j} = \begin{cases} 
\Delta^j A_{pq+r+j} + (\Delta^j - 1)(A_{r+j} + A_{p+r+j} + \cdots + A_{p(q-1)+r+j}) & \text{if } r + j \leq p - 1, \\
\Delta^j (A_{pq+r+j} + A_{r+j-p}) + (\Delta^j - 1)(A_{r+j} + A_{p+r+j} + \cdots + A_{p(q-1)+r+j}) & \text{if } r + j \geq p.
\end{cases}
\]

Hence, \( \Gamma \) is weakly distance-regular and, from (13) and (14), its distance polynomials are easily derived. In particular, \( \Gamma \) is \( \Delta \)-regular.

Now, let us assume that \( \Gamma \) is a distance-regular digraph with degree \( \Delta > 1 \). Let us take a vertex \( u \in V_0 \) and let \( v \) and \( w \) be two distinct out-neighbours of \( u \); that is, \( v, w \in \Gamma^+_1(u) \subseteq V_1 \).

Taking into account that the adjacency matrix \( A \) of \( \Gamma \) is normal,

\[
|\Gamma^+_1(v) \cap \Gamma^+_1(w)| = (AA^\top)_{vw} = (A^\top A)_{vw} = |\Gamma^-_1(v) \cap \Gamma^-_1(w)| \geq 1,
\]

which means that there is a vertex \( z \in V_2 \) adjacent from \( v \) and \( w \). As a consequence, \( \Gamma \) has two \( u \to z \) walks of length two, which is impossible unless \( D - (p - 1) < 2 \); that is, \( D = p \).

Hence, \( \Gamma \) cannot be distance-regular unless \( \Delta = 1 \) (\( D = pq + p - 1 \)) or \( D = p \) (\( \Delta > 1 \)), which correspond to the cycle digraph \( C_{p(q+1)} \) and to the complete generalized \( p \)-cycle \( K^{GC}_{\Delta, \ldots, \Delta} \), respectively.

As we will see, the intersection matrix of a Moore generalized \( p \)-cycle of diameter \( D \) has the eigenvalue 0 that is not simple, if \( D - (p - 1) \equiv 0, 1 \pmod{p} \). To cope with this case we need to extend the applicability of (8).

**Proposition 3.2.** Let \( \Gamma \) be a weakly distance-regular digraph of diameter \( D \), order \( N \) and distance polynomials \( \{p_k\}_{k=0}^D \). Let \( B \) be the intersection matrix of \( \Gamma \) and let \( \lambda_i \) be an eigenvalue of \( B \). If \( \lambda_i \) has a left standard eigenvector \( u_i^\top \) then the multiplicity of \( \lambda_i \), as an eigenvalue of \( \Gamma \), is

\[
m(\lambda_i) = \frac{N}{u_i^\top v_i},
\]

where \( v_i = (p_0(\lambda_i), p_1(\lambda_i), \ldots, p_D(\lambda_i))^\top \).
Proof. Let \( \{\lambda_j\}_{j=0}^d \) be the distinct eigenvalues of \( B \), where \( 0 \leq d \leq D \), with algebraic multiplicities \( \{m(\lambda_j)\}_{j=0}^d \) as eigenvalues of \( \Gamma \). Let \( \{A_k\}_{k=0}^D \) be the distance matrices of \( \Gamma \) and let \( A \) be the adjacency matrix of \( \Gamma \). Then,

\[
\text{tr} A_k = \text{tr} p_k(A) = \sum_{j=0}^d m(\lambda_j)p_k(\lambda_j) = N\delta_{k0}, \quad (0 \leq k \leq D).
\]

Such a linear system can be written in matrix terms as follows

\[
Pm = Ne_0,
\]

where \( e_0 = (1, 0, \ldots, 0)^T, \ m = (m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_d))^T \) and \( P \) denotes the \( D \times d \) matrix whose \((k, j)\)-element is \( p_k(\lambda_j), 0 \leq k \leq D \) and \( 0 \leq j \leq d \). We know that the \( j \)th column of \( P \), \( v_j = (p_0(\lambda_j), p_1(\lambda_j), \ldots, p_D(\lambda_j))^T \), is a right standard eigenvector of \( B \) corresponding to the eigenvalue \( \lambda_j \). Let \( U \) be a matrix whose \( j \)th row, \( u_j^T \), is a left eigenvector of \( B \) corresponding to the eigenvalue \( \lambda_j, 0 \leq j \leq d \). We assume that the first component of \( u_j \) is 1, if \( j = i \). Since \( UPm = NUe_0 \), we have

\[
\sum_{j=0}^d (u_i^T v_j)m(\lambda_j) = N.
\]

Notice that \( u_i^T v_j = 0 \), if \( i \neq j \), since \( \lambda_i u_i^T v_j = u_i^T Bv_j = \lambda_j u_i^T v_j \). Therefore,

\[
(u_i^T v_i)m(\lambda_i) = N,
\]

which implies (8). \( \square \)

In [13] we determined the spectrum of a wrapped butterfly digraph by using the submatrix \( P_{ee} = (p_k(\lambda_j))_{0 \leq k, j \leq d} \).

**Theorem 3.3.** The spectrum of a Moore generalized \( p \)-cycle \( \Gamma \) of degree \( \Delta \) and diameter \( D \), where \( D - (p - 1) = pq + r \) and \( 0 \leq r \leq p - 1 \), is

\[
\text{sp} \Gamma = \left( \begin{array}{cccc}
\Delta \xi^j & \xi^2 \xi^j & \ldots & \xi^q \xi^j \\
1 & m(\xi) & m(\xi^2) & \ldots & m(\xi^q) & N(\Delta^r - 1) \\
\end{array} \right), \quad j = 0, 1, \ldots, p - 1, \quad (15)
\]

where \( \xi = e^{2\pi i/p} \) and \( \xi \) is a primitive \((D + 1 - r)\)th root of unity, say \( \xi = e^{2\pi i/(p(q+1))} \), and

\[
m(\xi^k) = N \frac{\Delta^{p-r}}{D + 1 - r} \frac{\xi^{pk} - 1}{\xi^k - \Delta^p}, \quad k = 1, 2, \ldots, q,
\]

where \( N = p\Delta^r(1 + \Delta^p + \ldots + \Delta^{pq}) \).

Moreover, the minimum polynomial of \( \Gamma \) is

\[
m_\Gamma = x^r(x^p - \Delta^p)\frac{x^{D+1-r} - 1}{x^{p-1}}. \quad (16)
\]

In particular, \( \lambda = 0 \) is an eigenvalue of \( \Gamma \) unless \( r = 0 \).
Proof. From Proposition 1.2 and using the distance polynomials of $\Gamma$, \( \{p_k\}_{k=0}^D \), given in Proposition 3.1, the minimum polynomial of $\Gamma$ is

\[
m_{\Gamma} = \Delta^{p-1} (x - \Delta) \sum_{k=0}^{D} p_k = x^r(x^p - \Delta^p) x^{D+1-r-1} / x^{p-1},
\]

where $\zeta$ is a primitive $p$th root of unity and $\xi$ is a primitive $(D + 1 - r)$th root of unity. Notice that, since $\Gamma$ is a connected generalized $p$-cycle, the whole spectrum of $\Gamma$, regarded as a system of points in the complex plane, goes over into itself under a rotation of the plane by the angle $2\pi / p$ (Perron-Frobenius Theorem; see [34]).

Besides, it can be checked that the non-null coefficients of the intersection matrix $B = (\xi_{ij})$ of $\Gamma$ are

\[
p_{ij} = \begin{cases} 
1 & \text{if } k = i + 1 \text{ and } i \leq D - p, \\
\Delta - 1 & \text{if } k \equiv i + 1 \mod p \text{ and } 0 < k < i + 1 \text{ and } D - (p - 1) \leq i \leq D, \\
\Delta & \text{if } k \equiv i + 1 \mod p \text{ and } k = 0, i + 1 \text{ and } D - (p - 1) \leq i \leq D. 
\end{cases}
\]

The standard right eigenvector of $B$ corresponding to $\xi$ is $v_\xi = (p_0(\xi), p_1(\xi), \ldots, p_D(\xi))^T$, where

\[
p_k(\xi) = \begin{cases} 
eq k & \text{if } k \leq D - r, \\
0 & \text{if } D - r + 1 \leq k \leq D \quad (r \geq 1).
\end{cases}
\]

It turns out that $B$ has also a (unique) standard left eigenvector corresponding to $\xi$, $u_\xi = (1, x_1, \ldots, x_D)$, where

\[
x_k = \frac{1}{\xi}(x_{k-1} + (\Delta - 1)x_{D-j}) = \frac{1}{\xi} \left( x_{k-1} + (\Delta - 1) \left( \frac{\xi}{\Delta} \right)^{j+1-r} \right), \quad \text{where } j \equiv r - k \mod p \text{ and } 0 \leq j \leq p - 1,
\]

whence,

\[
x_k = \frac{1}{\xi}x_k + (\Delta^r - 1) \frac{\xi^{k-1}}{\Delta^{p-q} - \xi^{p-q-1}} + \frac{\Delta^{p-q-1}}{\Delta^{p-1}} \frac{\xi^{k-1}}{\xi^{p-q} - \xi^{p-q-1}}, \quad \text{if } k \equiv 0 \mod p,
\]

and

\[
x_k = \begin{cases} 
\frac{1}{\xi}x_{k-j} + \frac{1}{\xi}(\Delta^j - 1) & \text{if } k \equiv j \mod p \text{ and } 1 \leq j \leq r, \\
\frac{1}{\xi}x_{k-j} + \frac{1}{\xi}(\Delta^j - 1) + \frac{1}{\xi}(\Delta^{j-r} - 1) \frac{\xi^{p-\Delta^p}}{\Delta^{p-1}} & \text{if } k \equiv j \mod p \text{ and } r + 1 \leq j \leq p - 1.
\end{cases}
\]

Taking into account that $\xi^{p(q+1)} = 1$,

\[
u_\xi^T v_\xi = \sum_{k=0}^{D-r} x_k \xi^k = \sum_{j=0}^{p-1} \sum_{l=0}^{q} x_{p+l} j \xi^{p+j} = p \sum_{l=0}^{q} x_{p+l} \xi^l = \frac{D + 1 - r \xi^p - \Delta^p}{\Delta^{p-1} - \xi^p - 1}.
\]
So, applying (8),

$$m(\xi) = \frac{N}{u_\xi v_\xi} = \frac{N}{D + \pi i/\Delta - 1} \xi^{\pi i/\Delta} - 1.$$ 

Such a formula can be extended to any other eigenvalue $\zeta^k$, $k = 1, 2, \ldots, q$. It only remains to compute the multiplicity of 0 as an eigenvalue of $\Gamma$, when $1 \leq r \leq p - 1$. It can be seen that the subspace of left eigenvectors of $B$ corresponding to 0 is generated by $u_0^\top = (x_0, x_1, \ldots, x_D)$, whose non-null components are $x_j = 1$, $j \equiv r - 1 \mod p$ and $j < D$, and $x_D = 1/(1 - \Delta)$. So, only when $r = 1$ we get a standard vector $u_0$, in which case

$$m(0) = \frac{N}{u_0 v_0} = \frac{N}{\Delta/(\Delta - 1)} = N - \frac{N}{\Delta},$$

since $v_0 = (1, 0, \ldots, 0, -1)^\top$. If $2 \leq r \leq p - 1$, then

$$m(0) = N - p - p \sum_{k=1}^q m(\xi^k) = N - p - (\frac{N}{\Delta} - p) = N - \frac{N}{\Delta}.$$

\hfill $\Box$

**Corollary 3.4** ([27]). Moore generalized $p$-cycles of degree $\Delta > 1$ and diameter $D$ do not exist unless $p \leq D \leq 3p - 2$.

**Proof.** Let $D - (p - 1) = pq + r$, where $0 \leq r \leq p - 1$. From the multiplicity of $\xi = e^{2\pi i/(D + 1 - r)}$ as an eigenvalue of a Moore generalized $p$-cycle of degree $\Delta$ and diameter $D$, given by Theorem 3.3, $(\xi^p - 1)/(\xi^p - \Delta^p)$ must be rational. So, if $\Delta > 1$, then $\xi^p = e^{2\pi i/(q + 1)}$ is rational and, consequently, $q = 0, 1$; that is, $p - 1 \leq D \leq 3p - 2$. Notice that the case $D = p - 1$ corresponds to the multigraph $C^\Delta_p$ obtained by replacing each arc of the cycle digraph $C_p$ by $\Delta > 1$ multiple arcs. \hfill $\Box$

**Corollary 3.5** ([21]). Moore bipartite digraphs of degree $\Delta > 1$ and diameter $D$ do not exist unless $2 \leq D \leq 4$.

We remark that the enumeration of Moore digraphs can also be derived from Corollary 3.4, taking $p = 1$. Nevertheless, we have decided to keep their study apart since it provides a simpler understanding of the techniques used.

Next, we study when Moore generalized $p$-cycles are line digraphs. First, notice that, from Theorem 3.3, the spectrum of a Moore generalized $p$-cycle $\Gamma$ of degree $\Delta$ and diameter $D$, where $D - (p - 1) = pq + r$ and $1 \leq r \leq p - 1$, is the same as the spectrum of a Moore generalized $p$-cycle $\Gamma'$ with equal degree and diameter $D - r$, apart from the multiplicity of the eigenvalue 0. Thus, $sp \Gamma = sp L'T'$, where $L'T'$ is the $r$th iterated line digraph of $\Gamma'$. In the bipartite case ($p = 2$), Fiol et al. [20] proved that the line digraph structure is inherent to any Moore bipartite digraph $\Gamma$ of diameter $D = 4$. Here, we extend such a structural property, when $p \geq 2$ and $2p \leq D \leq 3p - 2$, using the following result that somehow describes the “hereditary character” of the regular line digraph structure (see [25]):

**Proposition 3.6.** Let $\Gamma$ be a regular digraph of degree $\Delta$ and order $N$ and let $A$ be its adjacency matrix. If there exists an integer $r \geq 1$ satisfying the following conditions:

(i) $A^{r+1}$ is a $(0, 1)$-matrix;
(ii) \( \text{rank } A^r = N/\Delta^r \);
then \( \Gamma \) is an \( r \)-th iterated line digraph.

We point out that, in the regular case, the condition \( \text{rank } A^r = N/\Delta^r \) is equivalent to say that the associated digraph to the \((0, 1)\)-matrix \( A^r \) is a line digraph (see \([24]\)).

**Theorem 3.7.** Every Moore generalized \( p \)-cycle of diameter \( D = 2p - 1 + r \), where \( 1 \leq r \leq p - 1 \), is the \( r \)-th iterated line digraph of a Moore generalized \( p \)-cycle of diameter \( 2p - 1 \).

**Proof.** Let \( A \) be the adjacency matrix of a Moore (regular) generalized \( p \)-cycle \( \Gamma \) of diameter \( D = 2p - 1 + r \), where \( 1 \leq r \leq p - 1 \), and order \( \Delta \) \((\Delta > 1)\). Since the multiplicity of the factor \( x \) in the minimum polynomial of \( A \) is \( r \) (see \((16))\), \( \dim \ker A^r \) is equal to the algebraic multiplicity of \( 0 \) as an eigenvalue of \( \Gamma \). So, from \((15)\),

\[
\text{rank } A^r = N - m(0) = N/\Delta^r.
\]

Besides, \( A^{r+1} \) is a \((0, 1)\)-matrix, since \( A^{r+1} = A_{r+1} \ (r + 1 \leq D - (p - 1)) \). Hence, using Proposition 3.6, we conclude that \( \Gamma = L^r \Gamma' \). Since the line digraph operator preserves all cycle lengths, \( \Gamma' \) must be a \( p \)-generalized cycle (every cycle of \( \Gamma' \) must have length multiple of \( p \)). Moreover, we can assume that \( \Gamma' \) has no isolated vertices, whence \( \Gamma' \) must be a regular digraph with degree \( \Delta \). Taking into account the relationship between the diameters and orders of a regular digraph and those of its line digraph (see \([22])\), we derive that \( \Gamma' \) has diameter \( D - r = 2p - 1 \) and order \( M_{GC}(\Delta, D)/\Delta^r = p(1 + \Delta + \cdots + \Delta^{pq}) = M_{GC}(\Delta, D - r) \).

Hence \( \Gamma' \) is also a Moore generalized \( p \)-cycle.

**Corollary 3.8 ([20]).** Every Moore bipartite digraph of diameter four is the line digraph of a Moore bipartite digraph of diameter three.

Notice that the unique Moore generalized \( p \)-cycle with diameter \( D = p \) is also a line digraph, \( K_{\Delta, \ldots, \Delta} = LC_p^A \). Moreover, it turns out that the enumeration of generalized \( p \)-cycles \( \Gamma = (V_0 \cup \cdots \cup V_{p-1}, E) \) with degree \( \Delta > 1 \) and diameter \( D = p - 1 + r \), where \( 2 \leq r \leq p - 1 \), is equivalent to the search of all binary submatrices \( \{ A_{\alpha,\alpha+1} \}_{\alpha=0}^{p-1} \) of order \( \Delta^r \) such that

\[
A_{\alpha,\alpha+1}J = \Delta J \quad \text{and} \quad \prod_{j=0}^{r-1} A_{\alpha+j,\alpha+j+1} = J, \quad \alpha = 0, 1, \ldots, p - 1,
\]

where the coefficients of \( A_{\alpha,\alpha+1} \) represent the adjacencies from \( V_{\alpha} \) to \( V_{\alpha+1} \). Thus, if \( r \) divides \( p \), we can construct \( A_{\alpha,\alpha+1} \) as a circulant matrix with Hall polynomial\(^1\)

\[
\theta_{A_{\alpha,\alpha+1}}(x) = 1 + x^{\Delta(j)} + \cdots + x^{(\Delta-1)\Delta}$

where \( \alpha \equiv j \mod r \) and \( 0 \leq j \leq r - 1 \), providing a Moore generalized \( p \)-cycle of diameter \( D = p - 1 + r \), which is not a line digraph.

\(^1\)In a circulant matrix \( A \) each row (excluding the first one) is the previous row moved to the right 1 place, with wraparound. In such a case, \( A \) is uniquely determined by its first row \((a_0, a_1, \ldots, a_{n-1})\) that can be represented by the polynomial

\[
\theta_A(x) = \sum_{i=0}^{n-1} a_i x^i.
\]

which is called the Hall polynomial of \( A \) (see \([18]\)).

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4 Almost Moore digraphs

The nonexistence of digraphs with order equal to the Moore bound $M_D(\Delta, D)$, when $\Delta, D > 1$, motivated the problem of the existence of almost Moore digraphs, namely digraphs with out-degree $\Delta > 1$, diameter $D > 1$ and whose order $N$ misses such unattainable bound by just one vertex; that is $N = \Delta + \Delta^2 + \cdots + \Delta^D$ (see e.g. [5, 3]). Every almost Moore digraph $\Gamma$ has the property that for each vertex $v \in V(\Gamma)$ there exists only one vertex, denoted by $r(v)$ and called the repeat of $v$, such that there are exactly two $v \rightarrow r(v)$ walks of length less than or equal to $D$ (one of them must be of length $D$). If $r(v) = v$, which means that $v$ is contained in (exactly) one cycle of length $D$, $v$ is called a selfrepeat of $G$. Since Miller et al. [35] proved that such extremal digraphs must be regular, it turns out that the map $r$, which assigns to each vertex $v \in V(\Gamma)$ its repeat $r(v)$, is an automorphism of $\Gamma$ (see [2]).

Now we characterize the subclass of almost Moore digraphs that are weakly distance-regular.

Proposition 4.1. An almost Moore digraph $\Gamma$ of degree $\Delta > 1$ and diameter $D > 1$ is weakly distance-regular if and only if each vertex of $\Gamma$ is contained in exactly one cycle of length $D$. In such a case, the distance polynomials of $\Gamma$ are $p_k = x^k$, $k = 0, 1, \ldots, D - 1$, and $p_D = x^D - 1$. Moreover, an almost Moore digraph can never be distance-regular.

Proof. Let $A$ be the adjacency matrix of an almost Moore digraph $\Gamma$ and $P$ be the permutation matrix associated with the map $r$ of repeats of $\Gamma$ such that its $(i, j)$ entry is 1 iff $r(i) = j$. Let $\{A_k\}_{k=0}^D$ be the set of distance matrices of $\Gamma$. Then, $A_k = A^k$, for each $k = 0, 1, \ldots, D - 1$, and $A_D = A^D - P$, since there is a unique $u \rightarrow v$ walk of length $l \leq D$ ($l = \text{dist}(u, v)$) unless $v = r(u)$, in which case there is an extra walk of length $D$. Therefore, if $\Gamma$ is weakly distance-regular, then $P = A^D - A_D$ belongs to the adjacency algebra $A(\Gamma)$, which has the set of distance matrices of $\Gamma$ as one of its basis. As a consequence, $P$ is a linear combination of $\{A_k\}_{k=0}^D$, which implies that $P = A_0 = I$, since $P$ is a permutation matrix. Hence, each vertex $v \in V(\Gamma)$ is a selfrepeat ($r(v) = v$), which means that $v$ is contained in exactly one cycle of length $D$. In such a case, $A_D = A^D - I$; that is, the distance polynomial of $\Gamma$ of degree $D$ is $p_D = x^D - 1$. Clearly, the remaining distance polynomials of $\Gamma$ are $p_k = x^k$, for each $k = 0, 1, \ldots, D - 1$.

Since the girth $g$ of an almost Moore digraph $\Gamma$ with all selfrepeat vertices is $g = D$, in order to satisfy the stability property (4) it is necessary that $A_{D-1} = A_1^T$, which implies that $\Delta^{D-1} = \Delta$. As a consequence, $\Gamma$ cannot be distance-regular when $D > 2$. In the case $D = 2$, $\Gamma$ would be a symmetric digraph ($A = A^T$) and, consequently, each vertex $v \in V(\Gamma)$ would be included in $\Delta > 1$ cycles of length $D = 2$, which is impossible. \hfill \Box

Theorem 4.2. Let $\Gamma$ be an almost Moore digraph with degree $\Delta > 1$, diameter $D > 1$ and such that all its vertices are selfrepeats. Then, the spectrum of $\Gamma$ is

$$\text{sp} \Gamma = \begin{pmatrix} \Delta & \xi & \xi^2 & \cdots & \xi^{D-1} & 0 \\ 1 & m(\xi) & m(\xi^2) & \cdots & m(\xi^{D-1}) & \Delta^D - 1 \end{pmatrix},$$

where $\xi = e^{2\pi i/D}$ and

$$m(\xi^k) = \frac{N \xi^k - 1}{D \xi^k - \Delta}, \quad \text{where } N = \Delta + \Delta^2 + \cdots + \Delta^D.$$

Moreover, the minimum polynomial of $\Gamma$ is

$$m_\Gamma = x(x - \Delta)(1 + x + \cdots + x^{D-1}).$$

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Proof. From the distance polynomials of $\Gamma$, $p_k = x^k$, if $k < D$, and $p_D = x^D - 1$, the construction of the intersection matrix $B$ of $\Gamma$ is derived,

$$B = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & \\
0 & 0 & 1 & \ddots & \ddots & \\
\ddots & \ddots & \ddots & \ddots & \ddots & \\
1 & 0 & 0 & \ddots & \ddots & 1 \\
\Delta - 1 & \Delta - 1 & \Delta - 1 & \ddots & \ddots & \ddots
\end{pmatrix}.$$

Besides, the minimum polynomial of $\Gamma$ is

$$\det(x I - B) = (x - \Delta) \sum_{k=0}^{D} p_k = (x - \Delta) x (1 + x + \cdots + x^{D-1}) = (x - \Delta) x \prod_{k=1}^{D-1} (x - \xi^k),$$

where $\xi = e^{2\pi i / D}$.

In order to obtain the multiplicity $m(\xi)$ of the $D$th root of unity $\xi$, as an eigenvalue of $\Gamma$, we need to compute the left standard eigenvector $u_\xi^\top = (1, x_1, \ldots, x_D)$ of $B$ corresponding to $\xi$. Since the condition $u_\xi^\top B = \xi B$ is equivalent to the recurrence relation

$$x_k = \frac{1}{\xi} (x_{k-1} + \Delta - 1), \quad k = 1, 2, \ldots, D \quad \text{and} \quad x_0 := 1,$$

it can be checked that

$$x_k = \frac{1}{\xi^k} + (\Delta - 1) \frac{\xi^k - 1}{\xi - 1} \xi^k, \quad k = 0, 1, \ldots, D.$$

Therefore, since the right eigenvector $v_\xi$ of $B$ corresponding to $\xi$ is $v_\xi = (1, \xi, \ldots, \xi^{D-1}, 0)^\top$, since $p_D(\xi) = \xi^D - 1 = 0$,

$$u_\xi^\top v_\xi = \sum_{k=0}^{D-1} x_k \xi^k = \sum_{k=0}^{D-1} \left( 1 + (\Delta - 1) \frac{\xi^k - 1}{\xi - 1} \right) = D \frac{\xi - \Delta}{\xi - 1},$$

and, consequently,

$$m(\xi) = \frac{N}{u_\xi^\top v_\xi} = \frac{N}{D} \frac{\xi - 1}{\xi - \Delta},$$

where $N$ is the order of $\Gamma$. As before, this formula also holds for any other $D$th root of unity $\xi^k$, if $1 \leq k \leq D - 1$.

The multiplicity of the eigenvalue $\lambda = 0$ is derived taking into account that $u_0^\top = (1, 1, \ldots, 1, 1/(1 - \Delta))$ and $v_0 = (1, 0, \ldots, 0, -1)^\top$, which gives

$$m(0) = \frac{N}{u_0^\top v_0} = \frac{\Delta + \Delta^2 + \cdots + \Delta^D}{\Delta/(\Delta - 1)} = \Delta^D - 1.$$

From the knowledge of the spectrum of almost Moore digraphs with all selfrepeat vertices we derive their classification. We point out that their nonexistence for diameter $D > 2$ was firstly proved by Bosák [9] and, subsequently, Baskoro et al. [4] provided a shorter proof.
**Corollary 4.3.** Almost Moore digraphs with all selfrepeats do not exist unless the diameter is two, in which case there is a unique solution, namely the Kautz digraph.

**Proof.** Let $\Gamma$ be an almost Moore digraph of degree $\Delta > 1$ and diameter $D > 1$, such that all its vertices are selfrepeats. From Theorem 4.2, the multiplicity of the eigenvalue $\xi = e^{2\pi i / D}$ of $\Gamma$ is $m(\xi) = \frac{N}{\Delta} \frac{\xi - 1}{\xi - \Delta}$, where $N$ is the order of $G$. Therefore, $\xi = e^{2\pi i / D}$ is rational, which implies that $D = 2$.

Now, we consider the case of diameter $D = 2$, where the minimum polynomial of $\Gamma$ is $m_\Gamma = x(x - \Delta)(x + 1)$ and its spectrum is

$$\text{sp } \Gamma = \begin{pmatrix} \Delta & -1 & 0 \\ 1 & \Delta & \Delta^2 - 1 \end{pmatrix},$$

since $\xi = e^{\pi i} = -1$ and $m(\xi) = \frac{\Delta + \Delta^2 - 2}{1 + \Delta} = \Delta$. This is the spectrum of the line digraph of the complete digraph of order $\Delta + 1$, $LK_{\Delta+1}$, which is isomorphic to the Kautz digraph of diameter two and degree $\Delta$ (see [22]). Since the multiplicity of the factor $x$ in $m_\Gamma$ is 1,

$$\text{rank } A = N - m(0) = \Delta + 1 = \frac{N}{\Delta},$$

Hence, $\Gamma$ is a line digraph; that is, $\Gamma = L K_{\Delta+1}$ (this was independently proved in [24] and [39]).

Furthermore, $LK_{\Delta+1}$ is the only almost Moore digraph of diameter $D = 2$ and degree $\Delta > 2$ (see [23]).

**References**


